The wording of a proof

Hardy's second "elegant" proof

— the Pythagorean school's irrationality of √2

JANICE PADULA

One of the most interesting and important proofs in the history of mathematics is the Pythagorean school's proof of the "irrationality" of √2. After a brief look at G. H. Hardy's (1941) thoughts regarding it, two versions of the classic Pythagorean proof are examined and discussed, one written by an American professor (King, 1992) and the other by an Australian mathematician, author and lecturer (Arianrhod, 2003). A 16-year-old student of Vietnamese/Chinese background is asked to prioritise the versions — which version is easiest to understand?

Proofs are an important part of what makes mathematics what it is, yet the challenges they present for students are generally unstated. One of the aims of the new mathematics curriculum document due for implementation in 2006, the Victorian Essential Learning Standards, or VELS (VCAA, 2005) for years Preparatory to 10, reads as follows: "develop understanding of the role of mathematics in life, society and work; the role of mathematics in history; and mathematics as a discipline — its big ideas, history, aesthetics and philosophy" (Victorian Curriculum Assessment Authority, 2005, p. 5).

Algebra is one of the "big ideas" in mathematics, as is proof. Devlin (2001) laments the passing of the formal teaching of geometry and comments that it was the only class in the high school curriculum that exposed students to "the important concept of formal reasoning and mathematical proof". He writes that exposure to formal mathematical reasoning is important because "a citizen in today's mathematically based world should have at least a general sense of one of the major contributors to society" (p. 78). Also, a survey by the United States Department of Education (The Riley Report, 1997) showed that students who completed high school geometry performed markedly better in gaining entrance to college (university) and did better at college than those students who had not taken geometry, regardless of the subjects studied at college (Devlin, 2001).

The mathematician G. H. Hardy (1941) thought that two proofs were simple, yet "elegant" and of the highest class. One was Euclid's proof of the existence of an infinity of prime numbers (Padula, 2003) and the other the Pythagorean school's proof of the "irrationality" of √2. Hardy thought that a theorem showed depth and generality if it were capable of considerable extension and was typical of a whole class of theorems of its kind. The relations shown by the proof should be such that they connect many different mathematical ideas.

Mathematical philosophy

Hardy's (1941) version of Pythagoras' proof is interesting because it is beautifully and concisely written and is based on classic logic — *reductio ad absurdum* — like Euclid's proof of the infinity of primes. Hardy, in a footnote, states that the proof can be arranged to avoid a reductio and that some logicians prefer not to use this form of argument. This is a reference to developments in logic such as that of the intuitionist school. (Hardy himself belonged to the realist, or "mathematics is discovered" school and believed that mathematical reality lies outside us and that the theorems that we prove, and which are described rather grandly as our "creations", are simply our notes of our observations. Devlin, 2001, holds a similar view.)

Since Kurt Gödel, an intuitionist, showed that in any system rich enough to express arithmetic there will be sentences proved which are false, or unprovable sentences which are true, perhaps classical logic has less currency than it once did, but because students should be aware of the importance of mathematics to society and the historical impact of this particular proof, it is good to marvel at the cleverness of the ancient Greeks and to share one's enthusiasm.

Sriraman (2003, 2004) found that students as young as 13–14 years were evenly divided between the Platonist, or realist (mathematics is discovered, mind independent) camp and the Formalist view, that is: mathematics is invented, or mind dependent. His students showed that they could discuss mathematical philosophy at an elementary level, after reading and discussing *Flatland* (Abbott, 1932) and some of the mathematical ideas in the first five chapters of Stewart's *Flatterland* (2001). It is worth noting here that Stewart (1996) argues that mathematics is neither invented nor discovered: it is a bit of both because "neither word adequately describes the process" and what mathematicians do is neither invention nor discovery, but a "complex context-dependent mix of both".

So what are the challenges for students when studying a proof and where will suitable versions of this proof be found?

The problem

Typically, proofs are embedded in language. Ordinary language, in this case English, can be difficult to comprehend; there are different registers (most of us use more than one) and words can have different meanings in different contexts, fields of study and so on. Also, a seemingly simple sentence in "ordinary" English can have a complex inner structure that is difficult to process — as the study of linguistics (Devlin, 2001), psycholinguistics, and mathematics education (Padula, Lam & Schmidtke, 2001; 2002) shows. Mathematics is a highly symbolic language that often requires a whole hierarchy of previous knowledge for understanding. It follows that a proof written in a combination of two complex languages, English and mathematics (the English mathematical register together with mathematical sentences in the form of equations, and sometimes "ordinary" English as well), can be quite challenging for many students, not just students of non-English-speaking background.

Mathematics as language

Somewhat paradoxically, mathematical symbols combined with words can convey complex, powerful ideas more efficiently than everyday, or even literary, language. Arianrhod (2003) states that the symbolism of the language of mathematics is an extremely important and integral part of its content. When you think in terms of mathematical symbols as well as words, thought itself is economised because "the symbolism enables you to see at a glance patterns and generalities, similarities and differences, which may not be obvious if you think only in words" (ibid, p. 133). Mathematics is not merely descriptive, Arianrhod explains, its linguistic structure seems to reflect hidden physical structure. She illustrates her point by saying that Einstein's famous equation $E = mc^2$ (energy equals mass, times the speed of light squared) came first, and only later did experimental physicists discover that it described reality. Furthermore, she explains, the same kind of event occurred when James Clerk Maxwell expressed Faraday's ideas about electromagnetism mathematically, thus paving the way for Einstein's theory of relativity (1905).

Mathematics: A definition

Hardy (1941) states that the beauty of mathematics resides in the fact that mathematics is all about, not just patterns, but patterns of ideas. Devlin defines mathematics as the "science of patterns" and then more fully as: "the science of order, patterns, structure, and logical relationships" (Devlin, 2001, p. 73). Mathematics has also often been described as the language of science. Since the mixture of symbols and words is so powerfully descriptive and communicative perhaps the definition: "mathematics is the language of the science of order, patterns, structure and logical relationships" may be considered.

Hardy's version

Hardy (1941) claims his version of the Pythagoreans' proof can be mastered in an hour by any intelligent reader however slender his "mathematical equipment". Obviously, that mathematical equipment must include some knowledge of number (theory) and algebra. Hardy uses the mathematical register of English; for example, he uses "integral" in the mathematical-adjectival sense, that is "whole number, integer", not the ordinary-English sense of "necessary to the completeness of the whole".

Hardy's language is quite sophisticated. Instead of merely stating the initial premise that the √2 is rational in an equation such as

$$\sqrt{2} = \frac{a}{b}$$

(where a and b are integers), as other writers have, Hardy writes:

> To say that "√2 is irrational" is merely another way of saying that 2 cannot be expressed in the form
>
> $$\left(\frac{a}{b}\right)^2;$$
>
> and this is the same as saying that the equation... $a^2 = 2b^2$ cannot be satisfied by integral values of a and b which have no common factor. (Hardy, 1941, p. 34)

(Please see the appendix for the full text of Hardy's version.)

Perhaps not surprisingly, "Linh", a 16-year-old Year-11 student of Vietnamese/ Chinese background at an Australian high school, thought that Hardy's version is "to (sic) intellectual to understand". (Linh has been doing mainstream English since Year 8; however, she was enrolled in English-as-a-Second-Language classes last year, her Year 12, as she was eligible.)

Now let us look at two other, hopefully simpler, versions.

King's version

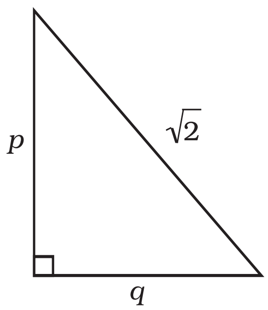
King (1992), mathematics professor, researcher and poet, includes in his book a version of Pythagoras' proof, as follows:

> Theorem: √2 is irrational.
>
> Proof: Suppose the theorem is false; i.e., suppose √2 is rational. Then we may write
>
> $$\sqrt{2} = \frac{p}{q}$$
>
> where p and q have no common factors. (Any original common factors may be canceled leaving numerator and denominator free of them.) Hence
>
> $$\sqrt{2}q = p$$
>
> or
>
> $$2q^2 = p^2.$$
>
> Thus, p^2 is an even number. It then follows that p is an even number. (It is easy to see that the square of any odd number is odd.) Thus, $p = 2c$ for some integer c. Therefore,
>
> $$2q^2 = (2c)^2$$
>
> or
>
> $$q^2 = 2c^2.$$
>
> Thus, q^2 is even and, just as before, it follows that q is even. Consequently, both p and q are even numbers and so are both divisible by 2. This contradicts our assumption that p and q have no common factors. So our hypothesis that √2 is rational is false. So, √2 is irrational and the theorem is proved.
>
> (King, 1992, pp. 136–137)

Note that King presumes a previous knowledge of rational and irrational numbers, and integers; they are discussed in the lead-up to this version of the proof in his book. In the beginning he has included equations Hardy left out, but, after placing $p = 2c$ within the

discourse he omits two "steps" or equations after $2q^2 = (2c)^2$. These are: $2q^2 = p^2 = (2c)^2 = 4c^2$, and $2q^2 = 4c^2$.

Linh, after studying King's version, made it her "No. 2 priority", stating that it is suitable for "people who likes (sic) numeracy explanation (sic)". She demonstrated her understanding of it geometrically with a diagram of a right-angled triangle with its sides labelled p and q, and the hypotenuse $\sqrt{2}$. She placed in the margin a mnemonic, the equation m = rise/run (the gradient, m, equals the vertical axis, or "rise", divided by the horizontal axis, or "run"). She included the equation $2q^2 = 4c^2$ that King had left out, and wrote the mathematical symbol for "therefore", ∴, in the appropriate place before $q^2 = 2c^2$.



Use $m = \dfrac{\text{rise}}{\text{run}}$ $\qquad \sqrt{2} = \dfrac{p}{q}$

Figure 1. Linh's diagram of her understanding of King's (1992) version of the proof.

Arianrhod (2003), in an appendix, has provided a version of the proof for the "general reader". She takes little for granted and has embedded the proof in an explanatory narrative of ordinary language, although she still uses vocabulary from the English mathematical register. Words such as "denominator" and "ratio" that are unexplained in the proof have usually been described in the main body of her book.

Arianrhod's version

To prove that $\sqrt{2}$ is irrational, all you do is explore what would happen if it were rational that is, if it could be written as

$$\sqrt{2} = \frac{x}{y}$$

where 'x' and 'y' are positive whole numbers which give the simplest possible fractional version of $\sqrt{2}$. (For example, $\frac{1}{2}$ is simpler than $\frac{2}{4}$ although they are both equal to one-half.) The idea is to solve this equation,

$$\sqrt{2} = \frac{x}{y}$$

and find appropriate values of x and y. An equation is 'solved' if actual numbers are found for 'x' and 'y', so that the left-hand side of the equation equals the right-hand side. But how can you do this if you do not know what number $\sqrt{2}$ is?

The trick is to square the equation — that is, you square the terms on each side of the equals sign:

$$\left(\sqrt{2}\right)^2 = \left(\frac{x}{y}\right)^2$$

that way, you are not changing the meaning of the equation because you have altered each symbol in exactly the same way, but you are rewriting it in a form which gives you 2 on the left-hand side of the 'equals' sign since by definition, $(\sqrt{2})^2 = 2$; you do not have to worry about what numerical value $\sqrt{2}$ actually has.

To summarise this procedure: square both sides of the equation

$$\sqrt{2} = \frac{x}{y}$$

to get

$$\left(\sqrt{2}\right)^2 = \frac{x^2}{y^2} \quad \text{or} \quad 2 = \frac{x^2}{y^2}$$

You now want to see if you can find whole numbers 'x' and 'y' that 'solve' this equation. If you were doing this with a rational number, say $\sqrt{9}$, you would have:

$$\sqrt{9} = \frac{x}{y}$$

which you would then square to give

$$9 = \frac{x^2}{y^2}$$

If you let $x = 3$ and $y = 1$, the equation is solved:

$$9 = \frac{3^2}{1^2} = \frac{9}{1} = 9 \quad \left(\text{so } \sqrt{9} = \frac{3}{1} = 3\right)$$

Surprisingly,

$$2 = \frac{x^2}{y^2}$$

is quite different. Rearrange it by taking the y^2 from the denominator (multiply both sides of the equation by y^2)

so that the original equation

$$\sqrt{2} = \frac{x}{y}$$

is now completely equivalent to the equation $2y^2 = x^2$. This means that x^2 is an even number, because it equals a multiple of 2 (that is, $2y^2$). But since 'x' is a positive, whole number, it must also be even, if its square is even. The even numbers, 2, 4, 6 ... all have even squares (4, 16, 36...), while the odd numbers, 1, 3, 5 ... have odd squares (1, 9, 25 ...). So to keep this clear in your mind, replace the 'x' in the equation $2y^2 = x^2$ by an obviously even number, $2p$, where 'p' is another whole number (which is half of the original number 'x'). The equation now reads $2y^2 = (2p)^2$, or $2y^2 = 2p \times 2p = 4p^2$.
The equation

$$\sqrt{2} = \frac{x}{y}$$

can now be written as $2y^2 = 4p^2$, and so the common factor 2 on both sides of the equation can be cancelled out to give $y^2 = 2p^2$. Now we have the same argument for 'y' as we had for 'x': it is a positive whole number, and since its square is even, it is even. Thus, the original 'x' and 'y' have to both be even if they are to satisfy the equation

$$\sqrt{2} = \frac{x}{y}$$

But any rational fraction can be expressed in simplest form, without both numbers being even (or having common factors). Therefore, $\sqrt{2}$ is not a rational fraction — it cannot be expressed as a ratio of two whole numbers. QED. (Arianrhod, 2003, pp. 282–283)

As you can see Arianrhod's version is highly explanatory and explicit. It is salutary for teachers since it shows the extent of (just some) of the prior learning necessary to master the proof. It also illustrates why Hardy thought the proof had depth and generality: it links many different mathematical ideas, not the least of which are number theory, rational and irrational numbers, and algebra. It contains ordinary English and she explains every step in detail. Some good students may be annoyed by being told what they know well but Arianrhod's version may be quite suitable for: high school students, students who like to confirm every point of an argument as they study it, and students who sometimes fail to grasp an individual step or two of an argument without further explication. (Of course all students can be advised that it helps to read any proof several times and to write it down.)

Students' reactions

NESB student Linh made many common-sense comments about Arianrhod's version. Here are some of them, in her words:

- Use simple language so students don't tune out.
- Too many unnecessary words can be confusing.
- Explain first then write equations.
- Make layout clear by putting equations on a new line.
- A layout with words and numbers "mixed" together not only doesn't look nice but also makes (the) student easily tune out. (Linh, 16)

In pencilled notes, Linh simplified the language even further, in places suggesting clearer phrasing, and she crossed out the rather formal (and probably unfamiliar), "To summarise this procedure." She also inserted equations, and an example in equations and words that summed up a wordy statement. However, Linh made Arianrhod's version her No. 1 priority, with the proviso that it was for "people who likes (sic) wordy explanation (sic)."

Nella (23 years), an applied-science graduate, when shown Hardy's, King's and Arianrhod's versions of the proof declared that the equations were more illuminating than the discourse. When asked why she thought so she replied, "If you know the patterns, the way algebra works, you can see… The language adds an extra complication."

Teaching choices

Obviously, high-school students should have mastered algebra before they attempt to understand the proof (although Arianrhod's explanations are both a reminder of algebraic rules and a crash course in same). Also, definitions (or reminders) of "integers," "rational" and "irrational" numbers should be given to students with, or prior to, study of the proof. Teachers should be aware of their students' command of the English mathematical register; i.e., words like "numerator" and "denominator," and, if the version demands it, ordinary English as well.

It seems that as well as considering the wording of a proof (Padula, Lam & Schmidtke,

2002) and the complexity of the underlying structure of its sentences (Padula, Lam & Schmidtke, 2001; Padula, 2003), teachers should also consider the layout of the words and symbols, so that students do not "easily tune out." If an equation is described in words it should be followed by the equation in symbols. (If possible, the equation should be removed from the text and displayed on a new line.) If an equation is to be inferred from the words but not actually written down as an equation, as in Hardy's version, remember that this necessitates the use of another linguistic skill: making a valid inference.

History of mathematics

Teachers may try the "history of mathematics" approach by first relating the story of the proof, the impact it made on the early Greek philosophers and why it was kept secret by them for many years. There are many excellent books currently available which tell the story; Arianrhod's (2003) highly readable book is but one, and an Internet search can be rewarding for students. One educational site suitable for projects on the history of mathematics and science is that of the School of Mathematics and Statistics at the University of St Andrews, Scotland, at:
http://www-groups.dcs.st-and.ac.uk/~history

If the proof is to be given not to an individual but to a group of capable students, teachers may decide to distribute one or both of the King and Arianrhod versions for study for an agreed period of time, to be followed by a group discussion; or, depending on students' abilities and knowledge of English and its mathematical register, teachers may choose between Arianrhod's (2003) thorough, explicit and explanatory — but rather wordy — version, and King's (1992) concise, business-like one. If preferred, King's version may be amended by the addition of some, or all, of the equations he did not use. If teachers would rather challenge students a little, King's version can be distributed as written and students asked to insert the skipped equations.

Conclusion

The proof of the irrationality of $\sqrt{2}$ is a very important part of the history of mathematics and makes a good introduction to formal mathematical reasoning; students may be enriched by its challenges. (Students may be interested to know that Einstein found mathematics very difficult but he did not let that stop him (Devlin, 2001).)

Learning to understand a proof is a linguistic/mathematical exercise. Words often have different meanings; when they are combined with mathematical symbols it makes for a powerful mix — a mix so powerful it helps us to explain the world to ourselves. Unfortunately, that mix can also be more intimidating for a learner than (well-known) mathematical symbols alone. Although there is no clear-cut way of assessing understanding, this may be gleaned from students' satisfaction that they have understood the proof, their diagrams, comments, the insertion of equations in the appropriate places, and their willingness to discuss the inherent ideas.

Teachers should know their students well and know if and when they need a challenge, or some assistance. It is not the few simple algebraic equations that make the proof of the irrationality of $\sqrt{2}$ difficult — on the contrary, the algebraic symbols and equations seem to aid understanding — rather, it is the ideas embedded in the mix of words and equations, and the twists and turns they take. Students may feel more in control of their learning, like Linh seems to, and may gain a better understanding of the proof, if they are invited to compare versions, and can refer from one to another to clarify a point.

Hardy's second "elegant" proof, the Pythagorean school's proof of the irrationality of $\sqrt{2}$, has the potential to give students an insight into the world of pure mathematics and a glimpse into the history of mathematics. Ideally, it may also give students cause to think about the conceptual power of mathematics and help hone their skills in mathematical technique — something Hardy thought was of the utmost importance.

Appendix: Hardy's version of Pythagorean school's proof of the irrationality of √2

A 'rational number' is a fraction a/b, where a and b are integers: we may suppose that a and b have no common factor, since if they had we could remove it. To say that '√2 is irrational' is merely another way of saying that 2 cannot be expressed in the form

$$\left(\frac{a}{b}\right)^2$$

and this is the same as saying that the equation

$$a^2 = 2b^2 \tag{B}$$

cannot be satisfied by integral values of a and b which have no common factor. This is a theorem of pure arithmetic, which does not demand any knowledge of 'irrational numbers' or depend on any theory about their nature.

We argue … by *reductio ad absurdum*; we suppose that (B) is true, a and b being integers without any common factor. It follows from (B) that a^2 is even (since $2b^2$ is divisible by 2), and therefore that a is even (since the square of an odd number is odd). If a is even then

$$a = 2c \tag{C}$$

for some integral value of c; and therefore

$$2b^2 = a^2 = (2c)^2 = 4c^2$$

or

$$b^2 = 2c^2 \tag{D}$$

Hence b^2 is even, and therefore (for the same reason as before) b is even. That is to say, a and b are both even, and so have the common factor 2. This contradicts our hypothesis, and therefore the hypothesis is false. (Hardy, 1941, pp. 34–36)

Acknowledgement

With gratitude to Sucy Lam and Marie Schmidtke for their comments.

References

Abbott, E. (1932). *Flatland: A Romance of Many Dimensions (4th Ed. Rev.)*. Oxford: Basil Blackwell.

Arianrhod, R. (2003). *Einstein's Heroes: Imagining the World through the Language of Mathematics*. Brisbane: University of Queensland Press.

Devlin, K. (2001). *The Maths Gene: Why Everyone Has it, but Most People Don't Use It*. London: Orion Books.

Hardy, G. H. (1941). *A Mathematician's Apology*. Cambridge: University Press.

King, J. P. (1992). *The Art of Mathematics*. New York: Plenum Press.

Padula, J. (2003). An elegant proof. *Australian Senior Mathematics Journal, 17*(1), 30–33.

Padula, J., Lam, S. & Schmidtke, M. (2001). Syntax and word order: Important aspects of mathematical English. *The Australian Mathematics Teacher, 57*(4), 31–35.

Padula, J., Lam, S. & Schmidtke, M. (2002). Mathematical English: Some insights for teachers and students. *The Australian Mathematics Teacher, 58*(3), 40–44.

School of Mathematics and Statistics, University of St Andrews (2005). Indexes of Biographies. Retrieved 16 May 2006 from http://www-groups.dcs.st-and.ac.uk/~history/BiogIndex.html.

Sriraman, B. (2003). Mathematics and literature: Synonyms, antonyms or the perfect amalgam? *The Australian Mathematics Teacher, 59*(4), 26–31.

Sriraman, B. (2004). Mathematics and literature (the sequel): Imagination as a pathway to advanced mathematical ideas and philosophy. *The Australian Mathematics Teacher, 60*(1), 17–23.

Stewart, I. (1996, Nov.). Think maths. *New Scientist, 152*(2058), 38–42.

Stewart, I. (2001). *Flatterland: Like Flatland Only More So.* London: Macmillan.

Victorian Curriculum Assessment Authority (2005). *Victorian Essential Learning Standards — Mathematics.* Melbourne: State Government of Victoria. Retrieved 16 May 2006 from www.vcaa.vic.edu.au.

Janice Padula
Ivanoe, Vic.
janicepadula@yahoo.com